TURBULENT WALL MOTION OF A VISCOELASTIC FLUID

Ya. S. Khodorkovskii

UDC 532.135:532.517.4

Qualitative agreement between the theoretically calculated parameters and published experimental data is obtained for turbulent wall flow of a drag-reducing aqueous polymer solution treated as a viscoelastic medium.

It is hypothesized in the majority of studies of the anomalous properties of aqueous polymer solutions that the reduction of frictional drag in flow over rigid boundaries (Toms effect) is attributable to the viscoelastic attributes of those solutions. The stated hypothesis requires the use of rheological relations other than the Newtonian equations in the derivation of the equations of motion for the indicated solutions. The problem is complicated, however, by the lack of theoretical principles justifying the choice of an appropriate rheological relation for the description of the motion of drag-reducing polymer solutions.

Adopting an ultimately simple model of a viscoelastic fluid as the model of the indicated solutions, we use the Maxwell and Oldroyd equations [1] as the "competing" rheological relations. They are written in the following form for the case of a plane-parallel fluid flow in the direction of the x axis in the half-space y > 0:

a) Maxwell model:

$$\tau+\theta_1\frac{d\tau}{dt}=\mu\frac{\partial u}{\partial y}\,; \tag{1}$$

b) Oldroyd model:

$$\tau+\theta_1\frac{d\tau}{dt}=\mu\frac{\partial u}{\partial y}+\mu\theta_2\frac{d}{dt}\left(\frac{\partial u}{\partial y}\right). \tag{2}$$

Applying the Reynolds averaging operation to all terms of the momentum equation for a continuum with regard for relations (1) or (2), we obtain the average equation of turbulent motion of a viscoelastic fluid past an infinite rigid wall. Integration of this equation across the flow yields a relation between the tangential stresses at the wall as well as the average and fluctuation velocity fields of the turbulent flow.

For rectilinear motion in a circular tube of radius r the equation assumes the form

$$\mu\frac{\partial\bar{u}}{\partial y}-\rho\overline{u'v'}+\mu\theta_2\overline{v'\frac{\partial^2u'}{\partial y^2}}-\rho\theta_1\overline{(v')^2}\,\frac{\partial\bar{u}}{\partial y}-\rho\theta_1\overline{(v')^2\frac{\partial u'}{\partial y}}=\tau_0\left(1-\frac{y}{r}\right). \tag{3}$$

It is evident from (3) that allowance for the elastic properties of the medium results in additional turbulent stresses, the quantitative estimation of which requires, as for the Reynolds stresses ($-\rho\overline{u'v'}$), additional relations between the average and fluctuation velocity fields. Accordingly, we rely on the most commonly used mixing-path hypothesis in semiempirical turbulence theories, whereby

$$u'\sim v'\sim l\,\frac{\partial\bar{u}}{\partial y}\,. \tag{4}$$

The expression for the length l of the mixing path is fundamental to the remainder of the solution because not only the viscosity of the fluid, but also its elastic properties,

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 30, No. 1, pp. 109-114, January, 1976. Original article submitted February 18, 1975.

clearly have a significant effect on the variation of l across the flow.

The Van Driest equation is most commonly used to describe the length of the mixing path [2] in the theory of turbulent wall flow of a Newtonian fluid. The velocity profile obtained near the wall by means of this equation is in good agreement with the experimental results both in the viscous substrate and logarithmic layer, as well as in the transition region between them. The stated agreement is a consequence of the recognition of interaction between effects of a molecular and molar nature in the viscous substrate and transition region of the flow.

We generalize the notions of Van Driest to the case of turbulent motion of a viscoelastic fluid. To do so we first solve the auxiliary problem of the velocity distribution in a viscoelastic fluid next to an infinite plane surface executing simple harmonic vibrational motion according to the law $u = U_0 \cos \omega t$ in its own plane. Due to flow symmetry, $u = u(y, t)$, $v = 0$, and the equation of motion takes the form

$$\rho \frac{\partial u}{\partial t} = \frac{\partial \tau}{\partial y} . \tag{5}$$

Using the more general rheological relation (2), we obtain from (5)

$$\theta_1 \frac{\partial^2 u}{\partial t^2} - \nu\theta_2 \frac{\partial^3 u}{\partial t \partial y^2} + \frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial y^2} = 0. \tag{6}$$

For this equation we obtain a solution that reverts to $U_0 \cos \omega t$ at $y = 0$ and to zero at $y = \infty$ in the form

$$u = U_0 \exp(-Hy) \cos(\omega t - Gy), \tag{7}$$

where

$$H = \sqrt{\frac{\sqrt{(1+\gamma_1^2)(1+\gamma_2^2)} + (\gamma_2 - \gamma_1)}{1+\gamma_2^2}} \sqrt{\frac{\omega}{2\nu}} ,$$

$$G = \frac{1}{H} , \quad \gamma_1 = \theta_1\omega, \quad \gamma_2 = \theta_2\omega.$$

If we invert the dynamic problem such that the infinite plane wall is now at rest and the velocity u is treated as the fluctuation component of the turbulent flow velocity, we obtain for that component, neglecting the cosine factor,

$$u' = U_0' [1 - \exp(-Hy)], \tag{8}$$

where U_0', the velocity fluctuation in the logarithmic layer, is independent of the viscous and elastic properties, and is given by the following expression according to Prandtl:

$$\sqrt{\overline{(U_0')^2}} = ky \frac{\partial \bar{u}}{\partial y} , \tag{9}$$

where $k = 0.4$.

Assuming in (7) that the factor $\sqrt{\omega\nu}$ is proportional to the dynamic velocity u_τ, we obtain the following equation for the mixing length:

$$l = ky \left[1 - \exp\left(- \frac{\alpha\eta}{A} \right) \right] , \tag{10}$$

in which $\eta = yu_\tau/\nu$ and $A = 26$ (Van Driest constant). This equation differs from the analogous Van Driest equation by the fact that the exponent includes the factor α, which characterizes the elastic properties of the medium. It follows from (7) that

$$\alpha = \sqrt{\frac{\sqrt{\left[1+\left(\frac{2\beta_1}{A^2}\right)^2\right]\left[1+\left(\frac{2\beta_2}{A_2}\right)^2\right]} + \frac{2(\beta_2-\beta_1)}{A^2}}{1+\left(\frac{2\beta_2}{A^2}\right)^2}}, \tag{11}$$

where $\beta_i = \tau_0\theta_i/\mu$ $(i = 1, 2)$.

Using the generalized equation (10) for the mixing length, we obtain from (3)

$$a_1\varphi'''\varphi' + a_2\varphi''(\varphi')^2 + a_3\varphi''\varphi' + a_4(\varphi')^3 + a_5(\varphi')^2 + a_6\varphi' + a_6R(z-1) = 0, \tag{12}$$

where

$$a_1 = -\frac{\beta_2}{R}z^2Q^2, \quad a_2 = \beta_1 kz^3Q^3,$$

$$a_3 = -2\frac{\beta_2}{R}zQ\left[Q + \frac{zR\alpha}{A}(1-Q)\right],$$

$$a_4 = -\beta_1 z^2Q^2 + k\beta_1 z^2Q^2\left[Q + \frac{zR\alpha}{A}(1-Q)\right],$$

$$a_5 = zQ\left[zRQ - \beta_2\frac{(1-Q)\alpha}{A}\left(2 - \frac{zR\alpha}{A}\right)\right],$$

$$a_6 = \frac{1}{k^2}, \quad Q = 1 - \exp\left(-\frac{zR\alpha}{A}\right),$$

$$z = \frac{\eta}{R}, \quad R = \frac{ru_\tau}{\nu}, \quad \varphi = \frac{\bar{u}}{u_\tau},$$

and the prime denotes differentiation with respect to z.

The minus sign is chosen for the terms $\overline{v'\partial^2u'/\partial y^2}$ and $\overline{u'v'}$ on the basis of physical considerations relating to the transfer of the corresponding substation due to the convective action of the transverse velocity fluctuation v'. The term $\overline{(v')^2\partial u'/\partial y}$, generally speaking, has a nonzero value, and its sign is not obvious. We therefore retain this term in the calculations and vary its sign. Plus signs for the coefficient a_2 and the second component of a_4 correspond to $\overline{(v')^2\partial u'/\partial y} < 0$.

Integrating Eq. (12) numerically under the boundary conditions $\varphi(0)=0$; $\varphi'(0)=R$; $\varphi'(1)=0$ for several pairs of values of β_1 and β_2 with variation of the sign of a_2 and second component of a_4, we show that the distribution of the average turbulent flow velocities next to the wall in a viscoelastic fluid differs appreciably from the analogous distribution in a Newtonian fluid. With an increase in the parameters of the viscoelastic characteristics under the condition $\theta_1 \gg \theta_2$ the thickness of the viscoelastic sublayer and width of the transition region increase. As shown by calculations, however, this difference is primarily due to the influence of the viscoelastic properties on the mixing length. Taking the latter consideration into account, along with the uncertainty of the sign of the term $\overline{(v')^2\partial u'/\partial y}$, we solve the problem below for rectilinear motion in a circular tube in the "linear approximation."

We integrate the simplified equation

$$k^2z^2Q^2R(\varphi')^2 + \varphi' + R(z-1) = 0, \tag{13}$$

whence we obtain

$$\varphi = \int_0^1 \frac{2R(1-z)}{\sqrt{1+4k^2R^2Q^2z^2(1-z)}+1} \tag{14}$$

The viscoelastic properties of the medium are accounted for in Eq. (13) in accordance with the mixing-length equation (10). The coefficient of viscous friction and the Reynolds number are given by the expressions

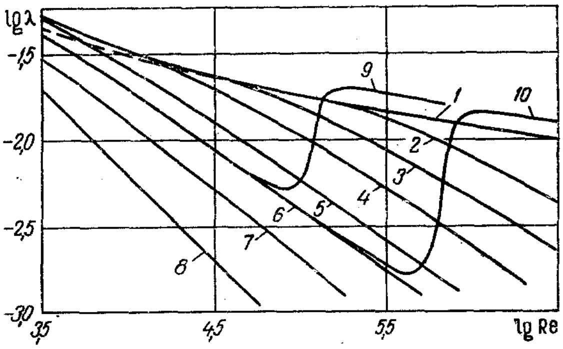

Fig. 1. Coefficient of viscous friction versus Reynolds numbers and elasticity. 1) $E_1 = E_2 = 0$ (Newtonian fluid), dashed curve according to the Nikuradze equation; 2-7) $E_2 = 0$ (Maxwell fluid); 2) $E_1 = 10^{-6}$; 3) 10^{-5}; 4) 10^{-4}; 5) 10^{-3}; 6) $5 \cdot 10^{-3}$; 7) $5 \cdot 10^{-2}$; 8) $\lambda = 64/\mathrm{Re}$ (laminar flow); 9) $E_1 = 5 \cdot 10^{-3}$, $E_2 = 5 \cdot 10^{-7}$; 10) $E_1 = 5 \cdot 10^{-3}$, $E_2 = 5 \cdot 10^{-9}$ (Oldroyd fluid).

$$\lambda = 8\left(\frac{u_\tau}{V}\right)^2, \quad \mathrm{Re} = 2\frac{V}{u_\tau}R, \tag{15}$$

in which

$$\frac{V}{u_\tau} = 2\int_0^1 \varphi(1-z)\,dz.$$

The calculations are carried out for constant values of the elastic coefficients E_i, which together with the Reynolds number are similarity criteria for the viscoelastic fluid and characterize the ratios of the relaxation (retardation) times θ_i and diffusion d^2/ν. The Weissenberg number W and Reynolds number Re are related to the elasticity criterion E by the expression

$$\mathrm{W} = E\,\mathrm{Re}. \tag{16}$$

The parameters β_i appearing in the mixing-length equation are related to the elastic constants E_i and Reynolds number Re by the expressions

$$\beta_i = \frac{1}{8}\lambda E_i\,\mathrm{Re}^2. \tag{17}$$

The results of the calculations for the Maxwell and Oldroyd models are given in Fig. 1. The results for the Maxwell model give better qualitative agreement with the published experimental data than those obtained for the Oldroyd model of a viscoelastic fluid. As the figure indicates, for $\theta_1 \gg \theta_2$ both models naturally yield consistent results over a reasonably broad range of Reynolds numbers.

The results illustrated in the figure satisfactorily account for many effects observed experimentally. For example, the given theory predicts the existence of a threshold value of the Reynolds number for the initiation of the drag-reduction effect, where for a constant concentration of the polymer solution the threshold Reynolds number increases with the diameter. Also, with an increase in the tube diameter, all other conditions being equal, the net drag reduction diminishes.

However, the theory fails to predict the experimentally observed saturation of the drag-reduction effect and its diminution with increasing Reynolds number for the case of dilute polymer solutions. It is well known that large shear stresses are accompanied by degradation of the polymer macromolecules or their formation into supramolecular structures. It is obvious that the parameters θ_1 and θ_2 differ in the degraded solution from the corresponding rheological parameters of the polymer solution at subcritical shear stresses.

We note in conclusion that Gorodtsov and Leonov [3] have used an alternative semiempirical theory of turbulent wall flow based on the model of a periodic viscous substrate to describe the motion of a viscoelastic fluid; their results are qualitatively consistent with those obtained in the present study.

NOTATION

u,v, longitudinal and transverse velocity components; u',v', fluctuation velocity components; l, mixing length; d, tube diameter; V, average (discharge) velocity in the tube; $u_\tau = \sqrt{\tau_0/\rho}$, dynamic velocity; μ, dynamic viscosity coefficient; ν, kinematic viscosity coefficient; θ_1, relaxation time; θ_2, retardation time; ρ, density; λ, coefficient of viscous friction; ω, cyclic frequency; $W = \theta V/d$, Weissenberg number; $E = \theta\nu/d^2$, elastic constant; $Re = Vd/\nu$, Reynolds number; ppm, weight concentration of the solution in polymer parts per million parts water.

LITERATURE CITED

1. F. R. Eirich (editor), Rheology: Theory and Applications, Academic Press (1956-1959).
2. E. R. Van Driest, J. Aeronaut. Sci., 23, No. 11, 1007 (1956).
3. V. A. Gorodtsov and A. I. Leonov, Inzh.-Fiz. Zh., 25, No. 6 (1973).

VISCOSITY OF BINARY LIQUID SYSTEMS

G. N. Dul'nev, Yu. P. Zarichnyak,
and B. G. Kapanadze

UDC 532.13

A method is proposed for calculating the viscosity of binary liquid systems (solutions, mixtures) involving the concentrations and properties of initial components on the basis of the theory of generalized conductivity.

Formulation of the Problem

A significant number of formulas for calculating the viscosity of binary liquid systems are known at present. These systems are obtained either by generalizing the experimental data (empirical formulas) or on the basis of the molecular theory of the liquid state.

We note that empirical formulas that describe the isotherms of viscosity of some single systems do not completely satisfy the experimental data for other systems. At times, empirical formulas satisfactorily describe only a part of the isotherm of a significant number of systems (in most cases, the area with the less viscous component), but these formulas do not agree with the experimental isotherm throughout its whole range [1]. Despite the inadequacies of empirical formulas, they do have a definite advantage because of their simplicity and reliability. To use formulas of the second group we must know the experimental values of the viscosity of a single mixture and, even better, of some of its compounds. In the latter case the accuracy of determining the viscosity according to these formulas significantly increases [2].

In calculating the coefficients of thermal conductivity and electric conductivity of binary systems, we can successfully use methods of the theory of generalized conductivity that are applied to the structure of a mixture with interpenetrating components. Below we show the possibility of extending this method to the calculation of the viscosity of binary liquid systems.

Institute of Mechanics and Optics, Leningrad. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 30, No. 1, pp. 115-121, January, 1976. Original article submitted April 21,1975